ADDITIVITY FOR BEGINNERS

ULI

Abstract. I recall parts of Florian's talk since a month has passed. Then I explain how operads become graded Lie algebras and operads with multiplication become Gerstenhaber algebras up to homotopy (which is an early instance of the $m = n = 1$ case of the additivity theorem).

Contents

1. Operads with multiplication

In this first section I recall and generalise various definitions introduced by Florian last time. Afterwards I define operads with multiplication, which is new material.

1.1. **Categorical setup.** Our aim is to study a specified type of algebraic structure, say a Lie algebra. This is an object X of some category $\mathscr{D}$ plus the algebraic structure on it, and the latter is given by the action of an operad. The operad itself is a functor O: $\mathscr{N}^{\mathrm{op}} \to \mathscr{C}$ whose domain $\mathscr{N}$ contains the "symmetries" used to describe the algebraic structure, while its values O(n) are the "spaces of operations" that constitute the algebraic structure itself, so $\mathscr{C}$ determines whether "space" means vector space, topological space, or something else. In many examples, $\mathscr{C}$ arises itself as a functor category $[\mathscr{I}, \mathscr{B}]$ where $\mathscr{B}$ is some cosmos one works in, and $\mathscr{I}$ is some index category (e.g. $\mathscr{B}$ vector spaces, $\mathscr{C}$ chain complexes).

category	role of objects
$\mathscr{D}$	carrier X of an O-algebra structure
$\mathscr{N}$	arity (shape of input) of an operation in O
$\mathscr{C}$	space O(n) of all n-ary operations in O
$\mathscr{B}$	anything that exists in our cosmos
$\mathscr{I}$	degree $\|\varphi\|$ of an operation $\varphi \in$ O(n)

1.1.1. *The category* $\mathscr{C}$. We make:

Assumption 1.1.1.1. Throughout this text, $(\mathscr{C}, \otimes, 1, \mathfrak{s})$ is a self-enriched symmetric monoidal category.

Example 1.1.1.2. We usually pretend $\mathscr{C}$ is one of the following:

(1) **Set** (sets, Florian just considered this case),
(2) **Top** (topological spaces) ⇝ “topological operads”,
(3) $\mathbf{Mod}_{\mathbb{K}}$ (vector spaces) ⇝ “algebraic operads”.

So we save time and space by working with elements rather than commutative diagrammes or in a graphical calculus.

Example 1.1.1.3. The symmetry will be represented by writing

$$\mathfrak{s}_{V,W}\colon V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w \otimes v,$$

even though the true formula could look different.

I also pretend throughout that all monoidal categories are strict and all (co)limits you see exist.

Example 1.1.1.4. A *graded vector space* is a sequence $V \in \mathscr{C} = \mathbf{GrMod}_{\mathbb{K}}$ of vector spaces $V(i)$, $i \in \mathbb{Z}$, and a morphism $V \to W$ is a sequence of morphisms $V(i) \to W(i)$ in $\mathbf{Mod}_{\mathbb{K}}$. The monoidal structure is given by

$$(V \otimes W)(n) \coloneqq \bigoplus_{i+j=n} V(i) \otimes W(j),$$

and the explicit formula for the symmetry is

$$\mathfrak{s}_{V,W}\colon V(i) \otimes W(j) \to W(j) \otimes V(i), \quad v \otimes w \mapsto (-1)^{ij} w \otimes v. \tag{1.1}$$

To make $\mathbf{GrMod}_{\mathbb{K}}$ self-enriched, we declare that all morphisms have degree $\mathbb{0}$,

$$\mathscr{C}(V, W)(n) \coloneqq \begin{cases} 0 & n \neq 0, \\ \prod_{i \in \mathbb{Z}} \mathbf{Mod}_{\mathbb{K}}(V(i), W(i)) & n = 0, \end{cases}$$

where 0 is the trivial vector space (the initial object in $\mathbf{Mod}_{\mathbb{K}}$). Note that $\mathscr{C}(-,-)$ is not an internal hom, that is, in general $\mathscr{C}(U \otimes V, W) \ncong \mathscr{C}(U, \mathscr{C}(V, W))$. However, $\mathscr{C}$ is closed with

$$\mathrm{hom}_{\mathscr{C}}(V, W)(j) = \mathscr{C}(V, s^j W),$$

where $s^j W \in \mathbf{GrMod}_{\mathbb{K}}$ is the j-fold *suspension* of W,

$$(sW)(n) \coloneqq W(n-1).$$

1.1.2. *The categories $\mathscr{I}$ and $\mathscr{B}$.* A large class of examples for $\mathscr{C}$ are functor categories $[\mathscr{I}, \mathscr{B}]$ where $\mathscr{B}$ is some base category and $\mathscr{I}$ is some index category. If you are happy with the examples $\mathscr{C} = \mathbf{Set}, \mathbf{Top}, \mathbf{Mod}_{\mathbb{K}}, \mathbf{GrMod}_{\mathbb{K}}$ just skip to 1.1.3 and take $\mathscr{I} = \mathbf{1}$, the terminal category with one object and one morphism, so that $\mathscr{B} = \mathscr{C}$.

Assumption 1.1.2.1. $(\mathscr{B}, \otimes, 1, \mathscr{t}, \mathrm{hom})$ is a *cosmos* (a symmetric closed monoidal category with all (co)limits). All categories occurring are from now on $\mathscr{B}$-enriched and all functors are $\mathscr{B}$-linear. $(\mathscr{I}, +, 0)$ is a (small) monoidal category.

Example 1.1.2.2. $\mathbf{Set}, \mathbf{Top}, \mathbf{Mod}_{\mathbb{K}}$ are default choices for $\mathscr{B}$.

Proposition 1.1.2.3. *The functor category $[\mathscr{I}, \mathscr{B}]$ carries a monoidal structure given by the coend*

$$(\mathsf{V} \otimes \mathsf{W})(n) := \sum^{(i,j)\in\mathscr{I}\times\mathscr{I}} \mathscr{I}(i+j, n) \otimes \mathsf{V}(i) \otimes \mathsf{W}(j). \tag{1.2}$$

Definition 1.1.2.4. $\mathsf{V} \otimes \mathsf{W}$ is the *Day convolution* of V and W.

Example 1.1.2.5. When $\mathscr{I}$ is the *terminal* category $\mathbf{1}$ (one object $0 \in \mathbf{1}$ with endomorphism object $\mathbf{1}(0,0) := 1 \in \mathscr{B}$, then $[\mathbf{1}, \mathscr{B}] = \mathscr{B}$, so one may always choose $\mathscr{C}$ to be the entire cosmos one is working in.

Example 1.1.2.6. To obtain $\mathbf{GrMod}_{\mathbb{K}}$, take $\mathscr{B} = \mathbf{Mod}_{\mathbb{K}}$ and $\mathscr{I} = \mathbb{Z}$, viewed as a discrete $\mathscr{B}$-enriched category,

$$\mathbb{Z}(i,j) := \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

where $1 = \mathbb{K}$ (the unit object in $\mathscr{B} = \mathbf{Mod}_{\mathbb{K}}$).

MSc Topic 1.1.2.7. Note that $[\mathscr{I}, \mathscr{B}]$ inherits a symmetry from $\mathscr{B}$, but the one given in (1.1) on $\mathbf{GrMod}_{\mathbb{K}}$ is more complicated and involves the choice of a natural action of $\mathscr{I}$ on $\mathbf{Mod}_{\mathbb{K}}$ given by the parity operator: there is a natural involution $\mathscr{p}$ on $[\mathbb{Z}, \mathscr{B}]$ given by $\mathscr{p}_{\mathsf{W}}(w) = (-1)^j w$ for $w \in \mathsf{W}(j)$, and this yields an action $i \triangleright w = \mathscr{p}^i_{\mathsf{W}}(w)$ of $\mathbb{Z}$. Now $\mathscr{s}_{\mathsf{V},\mathsf{W}}$ is given by $v \otimes w \mapsto v_{(-1)} \triangleright w \otimes v_{(0)}$ where $v_{(-1)} \otimes v_{(0)} = i \otimes v$ if $v \in \mathsf{V}(i)$.

Example 1.1.2.8. To obtain the category $\mathbf{Ch}_{\mathbb{K}}$ of chain complexes of vector spaces, one considers $\mathscr{B} = \mathbf{Mod}_{\mathbb{K}}$ and takes as $\mathscr{I}$ the category with object set $\mathbb{Z}$ and morphisms

$$\mathscr{I}(i,j) := \begin{cases} 1 & i = j, j-1, \\ 0 & i \neq j, \end{cases}$$

where $\mathscr{I}(i,i) = 1 = \mathbb{K}$ has a vector space basis given by id_i while $\mathscr{I}(i,i-1) = \mathbb{K}$ has a basis given by a morphism d_i for which $d_{i-1} \circ d_i = 0 \rightsquigarrow$ "DG (differentially graded) operads".

MSc Topic 1.1.2.9. One could generalise (1.2) to

$$(\mathsf{V} \otimes \mathsf{W})(n) := \sum^{(i,j) \in \mathscr{I} \times \mathscr{I}} \mathscr{I}(i+j,n) \diamond \mathsf{V}(i) \otimes \mathsf{W}(j), \tag{1.3}$$

assuming now that $\mathscr{I}$ is $\mathscr{A}$-enriched rather than $\mathscr{B}$-enriched and $\mathscr{B}$ is an $\mathscr{A}$-$\mathscr{B}$-bimodule category, $\mathscr{A}$ being yet another monoidal category that acts on $\mathscr{B}$ from the left via

$$\diamond \colon \mathscr{A} \times \mathscr{B} \to \mathscr{B}.$$

A good setting for Hebig's proof?

Example 1.1.2.10. Here is a cute example: Take $\mathscr{A} = \mathbf{Set}$, so that $\mathscr{I}$ is just any unenriched monoidal category, and observe that a cosmos $\mathscr{B}$ is anyway implicitly a $\mathbf{Set}$-$\mathscr{B}$-bimodule category: just define $A \diamond X$ to be the coproduct $\bigoplus_A X$, a direct sum of copies of $X \in \mathscr{B}$ indexed by the elements of $A \in \mathbf{Set}$ (exists as we assumed all colimits in $\mathscr{B}$ exist). That this commutes with the right action of $\mathscr{B}$ on itself follows as $\otimes$ is cocontinuous (since $\mathscr{B}$ is assumed to be closed). The fact that the coproduct has the universal property that it has and is not just any $\mathscr{A}$-module category means that

$$\underline{\mathscr{B}}(A \diamond X, Y) \cong \mathbf{Set}(A, \underline{\mathscr{B}}(X,Y)),$$

where $\underline{\mathscr{B}}(-.-) \colon \mathscr{B}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$ takes the set of morphisms in $\mathscr{B}$ (forget the self-enrichment).

1.1.3. *The category* $\mathscr{D}$. Our main aim is to define what it means to add an algebraic structure to an object of some category $\mathscr{D}$. The setup for this is precisely the one from 1.1.2.10, just "one level up":

Assumption 1.1.3.1. $(\mathscr{D}, \star, I)$ is a $\mathscr{C}$-enriched monoidal category and a $\mathscr{C}$-$\mathscr{D}$-bimodule category via a functor

$$\cdot \colon \mathscr{C} \times \mathscr{D} \to \mathscr{D},$$

that is, for $C \in \mathscr{C}, X, Y \in \mathscr{D}$, we have

$$(C \cdot X) \star Y \cong C \cdot (X \star Y).$$

Furthermore, $- \cdot X$ is left adjoint to $\mathscr{D}(X, -)$, so in $\mathscr{C}$, we have

$$\mathscr{D}(C \cdot X, Y) \cong \mathscr{C}(C, \mathscr{D}(X, Y)), \quad C \in \mathscr{C}, X, Y \in \mathscr{D}.$$

Remark 1.1.3.2. Some of the approaches to operads described below work in greater generality, I assume this in order to make them all equivalent.

Example 1.1.3.3. If $\mathscr{C} = \mathbf{Mod}_{\mathbb{K}}$, we could take $\mathscr{D} = \mathbf{Mod}_{R^e}$, where R is a $\mathbb{K}$-algebra (a monoid in $\mathscr{C}$) and $R^e = R \otimes R^{\mathrm{op}}$ is its *enveloping algebra* so that $\mathscr{D}$ is the category of R-bimodules with symmetric action of $\mathbb{K}$. Here

$$C \cdot X = C \otimes X,$$

where $\otimes = \otimes_{\mathbb{K}}$ is the tensor product of vector spaces (with left and right R-action just on X).

1.1.4. *The category $\mathcal{N}$.* The final ingredient we fix is a category that controls the symmetries of the monoidal structures $\otimes$ of $\mathscr{C}$ and $\star$ of $\mathscr{D}$. This can be generalised much further, but the following is sufficient to give you an indication how the various types of operads that occur in the literature can be treated in a unified way. I have cobbled this together from various sources and hope this is consistent.

Assumption 1.1.4.1. $(\mathcal{N}, +, 0)$ is a monoidal subcategory of the skeleton $\mathbb{F}$ on $\mathbf{FinSet}$ which has as objects the sets $i \coloneqq \{0, \ldots, i-1\}$, $i \in \mathbb{N}$, and as morphisms all set maps,

$$\mathbb{F}(i, j) \coloneqq \mathbf{Set}(i, j).$$

We consider $\mathbb{F}$ as a symmetric monoidal category with symmetric monoidal structure given by addition (concatenation).

Example 1.1.4.2. By $\mathbb{S} \subset \mathbb{F}$, I denote the *permutation category* which is the core of $\mathbb{F}$,

$$\mathbb{S}(i, j) \coloneqq \begin{cases} \varnothing & i \neq j, \\ S_i & i = j, \end{cases}$$

where S_i is the group of all permutations of $\{0, \ldots, i-1\}$.

Example 1.1.4.3. The natural numbers $\mathbb{N}$ are always treated as the discrete category with no morphism $i \to j$ if $i \neq j$ and one morphism $i \to i$, the identity.

Definition 1.1.4.4. An $\mathcal{N}$-*monoidal* structure on $\mathcal{D}$ assigns to $f \in \mathcal{N}(i,j)$ and $\mathsf{V}_0, \dots, \mathsf{V}_{j-1} \in \mathcal{D}$ a natural morphism

$$\mathfrak{s}^f_{\mathsf{V}_0,\dots,\mathsf{V}_{j-1}} \colon \mathsf{V}_0 \star \cdots \star \mathsf{V}_{j-1} \to \mathsf{V}_{f(0)} \star \cdots \star \mathsf{V}_{f(i-1)},$$

and if $g \in \mathcal{N}(k,i), h \in \mathcal{N}(m,n)$, we have

$$\begin{aligned} \mathfrak{s}^{f \circ g}_{\mathsf{V}_0,\dots,\mathsf{V}_{j-1}} &= \mathfrak{s}^{g}_{\mathsf{V}_{f(0)},\dots,\mathsf{V}_{f(i-1)}} \circ \mathfrak{s}^{f}_{\mathsf{V}_0,\dots,\mathsf{V}_{j-1}}, \\ \mathfrak{s}^{f+g}_{\mathsf{V}_0,\dots,\mathsf{V}_{j+n-1}} &= \mathfrak{s}^{f}_{\mathsf{V}_0,\dots,\mathsf{V}_{j-1}} \star \mathfrak{s}^{h}_{\mathsf{V}_j,\dots,\mathsf{V}_{j+n-1}}. \end{aligned}$$

Example 1.1.4.5. An $\mathbb{N}$-monoidal category is a plain monoidal category $\rightsquigarrow$ "planar (= nonsymmetric) operads".

Example 1.1.4.6. An $\mathbb{S}$-monoidal category is a symmetric monoidal category $\rightsquigarrow$ "symmetric operads".

Example 1.1.4.7. A little thought tells you that an $\mathbb{F}$-monoidal category is the same as a cartesian category: a symmetric monoidal category is cartesian if and only if every object is in a unique way a coalgebra and all morphisms are coalgebra morphisms. In terms of the $\mathbb{F}$-monoidal structure, the coproduct ("universal copying") is the morphism $\mathfrak{s}^m_{\mathsf{V}} \colon \mathsf{V} \to \mathsf{V} \star \mathsf{V}$, where $m \colon 2 \to 1$ maps both $0, 1$ to 0, and the counit ("universal deletion") is $\mathfrak{s}^e_{\mathsf{V}} \colon \mathsf{V} \to \mathrm{I}$, where $e \colon 0 \to 1$ is the initial map $\rightsquigarrow$ "cartesian operads (= clones = Lawvere theories)". See e.g. [12] for details.

Assumption 1.1.4.8. $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{N}$-monoidal categories.

Caveat 1.1.4.9. $\mathcal{C}$ is anyway assumed to be $\mathbb{S}$-monoidal, otherwise some of the definitions of an operad given below don't make sense. However, one still might choose $\mathcal{N} = \mathbb{N}$; then the operad O lives in a symmetric monoidal category $\mathcal{C}$ but one does not use this in the "defining equations" of O.

MSc Topic 1.1.4.10. This is by far not the most general setup that is studied, see e.g. [1, 2, 4, 8, 9, 10, 21] for some ideas where this is going. For starters, $\mathcal{N}$ could come with a monoidal functor $\mathsf{u} \colon \mathcal{N} \to \mathbb{F}$ rather than an inclusion; this assigns to a symmetry $f \in \mathcal{N}(x,y)$ an underlying map

$\mathsf{u}(f) \in \mathbb{F}(i, j)$, and the corresponding natural transformation is a morphism $\mathfrak{s}^{f}_{\mathsf{V}_0,\dots,\mathsf{V}_{j-1}} \colon \mathsf{V}_0 \star \cdots \star \mathsf{V}_{j-1} \to \mathsf{V}_{\mathsf{u}(f)(0)} \star \cdots \star \mathsf{V}_{\mathsf{u}(f)(i-1)}$.. This is needed e.g. when adding braided monoidal categories to the picture, where $\mathscr{N}$ is the braid category $\mathbb{B}$ which is like $\mathbb{S}$, but with the braid group B_i replacing S_i. The functor u is given by the group quotients $B_i \to S_i$. However, one can also consider situations in which the tensorands are not just j objects that are arranged in a linear order. So $\mathbb{F}$ could be replaced e.g. by categories of manifolds, see e.g. [3]. Once one generalises the setup in this way, one needs more conditions that we will mention briefly when we have enough material to explain them. In the classical approach of Kelly (Max, not Maggs), $\mathscr{N}$ should be a *club*. This defines a 2-monad $\mathfrak{K}_{\mathscr{N}}$ on $\mathbf{Cat}$ whose 2-algebras are $\mathscr{N}$-monoidal categories. In particular, $\mathscr{N}$ can be recovered as the free $\mathscr{N}$-monoidal category $\mathfrak{K}_{\mathscr{N}}(\mathbf{1})$ on a single generator.

MSc Topic 1.1.4.11. It might be sufficient to demand at least in some approaches that there is a functor $\mathscr{N} \times \mathscr{D} \to \mathscr{D}$, $(n, X) \mapsto X^{\star n}$ satisfying this or that instead of a fully fledged $\mathscr{N}$-monoidal structure.

MSc Topic 1.1.4.12. In which generality can we Jazz up $\mathbb{N}, \mathbb{S}, \mathbb{F}$ to make them $\mathscr{C}$-enriched?

1.2. **Operads.**

1.2.1. *Approach I and II.* I think this is the most elementary and immediate one, so I begin with this; the idea is to make sense of the following picture:

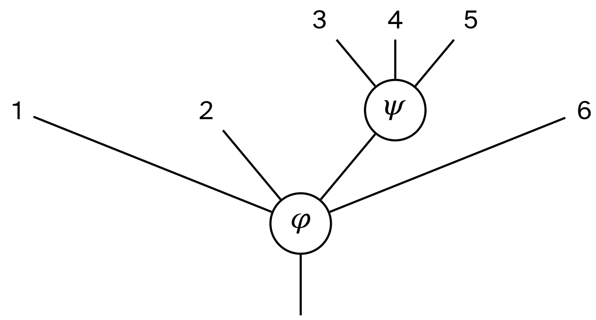


Figure 1. $\varphi \circ_{4,3} \psi$ ($p = 4, q = 3$)

Idea 1.2.1.1. An *($\mathcal{N}$-)operad (in $\mathscr{C}$)* is a functor

$$\mathsf{O}\colon \mathcal{N}^{\mathrm{op}} \to \mathscr{C}$$

together with morphisms

$$\circ_{p,i}\colon \mathsf{O}(p) \otimes \mathsf{O}(q) \to \mathsf{O}(p+q-1), \quad i = 1, \dots, p$$

satisfying a unitality axiom (there is a unary operation $\mathrm{id} \in \mathsf{O}(1)$ that is an identity for all $\circ_{p,i}$; just as with $\mathbb{K}$-algebra, some would drop this and consider also nonunital operads), and the associativity axiom

$$(\varphi \circ_{p,i} \psi) \circ_{p+q-1,j} \chi = \begin{cases} (\varphi \circ_{p,j} \chi) \circ_{p+r-1,i+r-1} \psi & j < i, \\ \varphi \circ_{p,i} (\psi \circ_{q,j-i+1} \chi) & i \leq j \leq i+q-1, \\ (\varphi \circ_{p,j-q+1} \chi) \circ_{p+r-1,i} \psi & i+q \leq j. \end{cases}$$

Furthermore, one requires the $\circ_{p,i}$ to be compatible with morphisms in $\mathcal{N}$, see 1.2.1.5 and 1.2.1.6 below.

Definition 1.2.1.2. We call $\varphi \in \mathsf{O}(n)$ an *n-ary operation in* O.

Remark 1.2.1.3. Note that in the associativity axiom, the symmetry $\mathfrak{s}$ of $\mathscr{C}$ enters where ψ and χ change places: just as you need a symmetric monoidal category to enrich monoidal categories, you need it to define operads. So when $\mathscr{C} = \mathbf{GrMod}_{\mathbb{K}}$ is the category of graded vector spaces, the above formulas suppress signs $(-1)^{|\psi||\chi|}$ in cases 1 and 3.

Caveat 1.2.1.4. Some people draw resp. read pictures upside down, some from right to left, some do both and then the indices in the associativity axiom change. Besides this convention on how to order the inputs of an n-ary operation, there is the convention on how to order the operations that are composed (whether Figure 1 depicts $\varphi \circ_{4,3} \psi$ or $\psi \circ_{4,3} \varphi$). In this, we stick to the traditional convention on compositions of functions. The abstract Approaches III - VI below avoid these troubles.

Equivalently (assuming unitality), one may define an operad as a functor $\mathsf{O}\colon \mathcal{N}^{\mathrm{op}} \to \mathscr{C}$ plus morphisms

$$\circ_{i_1,\dots,i_n}\colon \mathsf{O}(n) \otimes \mathsf{O}(i_1) \otimes \cdots \otimes \mathsf{O}(i_n) \to \mathsf{O}(i_1 + \cdots + i_n)$$

whose axioms are derived from pictures such as the following:

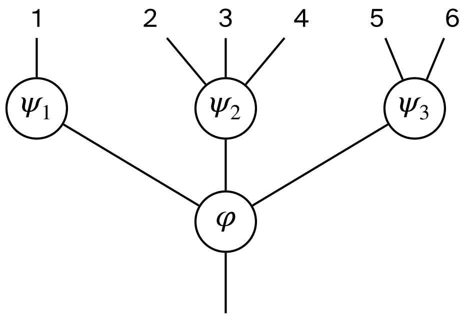


Figure 2. $\varphi \circ_{1,3,2} [\psi_1 \otimes \psi_2 \otimes \psi_3]$

The translation between Approaches I and II is given by

$$\varphi \circ_{p,i} \psi := \varphi \circ_{1,\ldots,1,q,1,\ldots,1} [\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \psi \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}]$$

respectively

$$\varphi \circ_{i_1,\ldots,i_n} [\psi_1 \otimes \cdots \otimes \psi_r] := (\varphi \circ_{p,1} \psi_1) \circ_{p+i_1-1,i_1+1} \psi_2 \otimes \cdots$$

Idea 1.2.1.5. Let us use Approach II to discuss the assumption that the composition operations in the operad are compatible with the symmetries prescribed by $\mathcal{N}$. There are two distinct topics to discuss, and we first consider symmetries acting on the ψ_r: for a morphism $f \in \mathcal{N}(p, q)$, we write $\mathsf{O}(f)$ as a right action,

$$\mathsf{O}(f) \colon \mathsf{O}(q) \to \mathsf{O}(p), \quad \varphi \mapsto \varphi \triangleleft f.$$

Figure 2 suggests to demand

$$\begin{aligned} &(\varphi \circ_{i_1,\ldots,i_n} [\psi_1 \otimes \cdots \otimes \psi_n]) \triangleleft (f_1 + \ldots + f_n) \\ &= \varphi \circ_{j_1,\ldots,j_n} [(\psi_1 \triangleleft f_1) \otimes \cdots \otimes (\psi_n \triangleleft f_n)] \end{aligned}$$

for all $f_r \in \mathcal{N}(j_r, i_r)$, $r = 1, \ldots, n$. Since we demanded that $\mathscr{s}^f \otimes \mathscr{s}^h = \mathscr{d}^{f+h}$, this can also be rewritten as

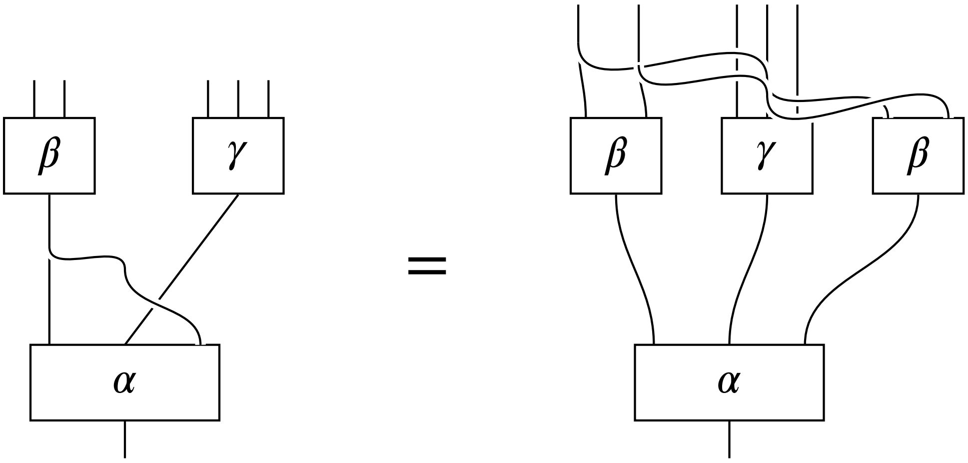
$$\begin{aligned} &(\varphi \circ_{i_1,\ldots,i_n} [\psi_1 \otimes \cdots \otimes \psi_n]) \triangleleft (f_1 + \ldots + f_n) \\ &= \varphi \circ_{j_1,\ldots,j_n} [(\psi_1 \otimes \cdots \otimes \psi_n) \triangleleft (f_1 + \cdots + f_n)]. \end{aligned}$$

So these symmetries simply tell us that the operadic composition is right $\mathcal{N}$-linear.

Idea 1.2.1.6. When it comes to symmetries acting on φ, things get a bit more interesting: Figure 2 also suggests to demand that for $g \in \mathcal{N}(m, n)$, we have

$$(\varphi \triangleleft g) \circ_{i_1,\ldots,i_m} [\psi_1 \otimes \cdots \otimes \psi_n] = \varphi \circ_{i_1,\ldots,i_m} g \triangleright [\psi_1 \otimes \cdots \otimes \psi_m],$$

where we must make sense of $g \triangleright [\psi_1 \otimes \cdots \otimes \psi_n]$, and we want that this can be expressed as $(\psi_{g(1)} \otimes \cdots \otimes \psi_{g(n)}) \triangleleft \hat{g}$ for some $\hat{g}$ that depends on g but also the arities of the ψ_i. Here is a picture that explains the situation; therein, we are considering $(\alpha \triangleleft g) \circ_{2,3} [\beta \otimes \gamma]$, where $\alpha, \gamma \in \mathrm{O}(3), \beta \in \mathrm{O}(2)$, and $g \in \mathbb{F}(3, 2)$ is given by $g(0) = g(2) = 0$, $g(1) = 1$.



The way how to get from g to $\hat{g}$ is what Kelly's club structure of $\mathcal{N}$ is about. At the end, this all means that we need the composition operation to be a morphism

$$\circ_{i_1,\dots,i_n} \colon \mathrm{O}(n) \otimes_{\mathcal{N}} [\mathrm{O}(i_1) \otimes \cdots \otimes \mathrm{O}(i_n)] \to \mathrm{O}(i_1 + \cdots + i_n).$$

MSc Topic 1.2.1.7. Write this all up in a Hopf algebraic language, $\mathcal{N}$ is the Hopf algebra, $\mathscr{C}$ is a bimodule? The above is somehow about the centre in a bimodule category.

Remark 1.2.1.8. Depending on one's taste one might prefer to give the inputs of operations *names* as in [13, Definition 16]. For example, you could replace $\mathbb{S}$ by the category of all totally ordered finite sets with all bijections as morphisms. Then Figure 2 shows $\varphi \circ_{1,3,2} [\psi_1 \otimes \psi_2 \otimes \psi_3] \in \mathrm{O}([1, 2, 3, 4, 5, 6])$, where [...] denotes an ordered set. If $g = (123) \in S_3$ is the cyclic permutation of the inputs of ψ, then the figure suggests that $(\varphi \triangleleft g) \circ_{3,2,1} [\psi_2 \otimes \psi_3 \otimes \psi_1] \in \mathrm{O}([2, 3, 4, 5, 6, 1])$ should be "the same" operation once one applies the appropriate block permutation of the 6 inputs. For this to make sense we need a functor $\mathrm{u} \colon \mathcal{N} \to \mathbb{S}$ as mentioned in 1.1.4.10 which tells us how to reorder the ψ_r in this process. Note that the input of an n-ary operation is not the object $\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_n$, it is the object with its decomposition into those factors. I think this is where one should think about slice categories and factorisation homology.

1.2.2. *Examples.*

Example 1.2.2.1. The *endomorphism operad* End_X of an object X in $\mathscr{D}$ has

$$\mathsf{End}_X(n) \coloneqq \mathscr{D}(X^{\star n}, X)$$

with

$$\varphi \circ_{p,i} \psi \coloneqq \varphi \circ (\mathrm{id}_{X^{\star i-1}} \star \psi \star \mathrm{id}_{X^{\star p-i}}).$$

For End_X to be of type $\mathscr{N}$, $\mathscr{D}$ must be a priori an $\mathscr{N}$-monoidal category. However, this is not an if and only if, it might happen that End_X has more symmetries than one expects. For example, in [12] we have shown that End_X can be naturally a cartesian operad even if $\mathscr{D}$ is not cartesian.

Definition 1.2.2.2. If O is any operad, then an O-*algebra structure* on X is an operad morphism $\alpha\colon \mathsf{O} \to \mathsf{End}_X$, that is, a natural transformation of functors $\mathscr{N}^{\mathrm{op}} \to \mathscr{C}$ that is compatible with the $\circ_{p,i}$.

Example 1.2.2.3. For $\mathscr{N} = \mathbb{N}$, the *planar associative operad* $\mathsf{Ass}^{\mathbb{N}}$ is given by setting for all n

$$\mathsf{Ass}^{\mathbb{N}}(n) \coloneqq 1,$$

with all $\circ_{p,i}$ being the canonical isomorphism $1 \otimes 1 \cong 1$. An $\mathsf{Ass}^{\mathbb{N}}$-algebra is a unital associative algebra (a monoid) in $\mathscr{D}$.

Example 1.2.2.4. When $\mathscr{N} = \mathbb{S}$ and we make all S_n act trivially on 1, we obtain the *commutative operad*

$$\mathsf{Comm}(n) \coloneqq 1.$$

That is, if we forget the trivial symmetry, the symmetric operad Comm becomes the planar associative operad. But we don't, and that a Comm-algebra structure $\alpha\colon \mathsf{Comm} \to \mathsf{End}_X$ is in particular a natural transformation of functors $\mathbb{S}^{\mathrm{op}} \to \mathscr{C}$ shows that the Comm-algebras are precisely the commutative algebras in $\mathscr{D}$; this structure could not have been defined when $\mathcal{N} = \mathbb{N}$. However, we may now define the *symmetric associative operad* $\mathsf{Ass}^{\mathbb{S}}$ by

$$\mathsf{Ass}^{\mathbb{S}}(n) := \mathsf{S}_n$$

and then $\mathsf{Ass}^{\mathbb{S}}$-algebras are again just algebras in $\mathscr{D}$. Note Florian was considering this symmetric associative operad, not the planar one.

Remark 1.2.2.5. There are also *cyclic* and *modular* operads, but this is about duality in $\mathscr{D}$, and about the spatial arrangement of the pictures that represent compositions that could be drawn not in the plane but on some oriented compact smooth manifold of dimension 2 (compare spherical and cylindrical monoidal categories). Here u is a functor to the category of 2-dimensional oriented surfaces I suppose.

MSc Topic 1.2.2.6. In the theory of simplicial sets, one can turn any set X into a simplicial set with $X_n = X$ and all simplicial operators being identities; this is the "discrete simplicial set", when passing to the geometric realisation you get the discrete topology on X. Comm and $\mathsf{Ass}^{\mathbb{N}}$ are like this. What does this mean? What happens for other $\mathcal{N}$?

1.2.3. *Approach III and IV.* Recall that we assume

$$\mathscr{D}(C \cdot X, Y) \cong \mathscr{C}(C, \mathscr{D}(X, Y)), \quad C \in \mathscr{C}, X, Y \in \mathscr{D}.$$

Idea 1.2.3.1. In a concrete setting, a sequence of morphisms

$$\alpha_n\colon C_n \to \mathscr{D}(X^{\star n}, X) = \mathsf{End}_X(n), \quad n \geq 0$$

in $\mathscr{C}$ thus corresponds to a sequence of morphisms

$$\hat{\alpha}_n\colon C_n \cdot X^{\star n} \to X$$

in $\mathscr{D}$. If $C_n = \mathsf{O}(n)$ for a functor $\mathsf{O}\colon \mathcal{N}^{\mathrm{op}} \to \mathscr{C}$ and the α_n are the components of a natural transformation, then the $\hat{\alpha}_n$ assemble into a single morphism $\hat{\alpha}\colon \hat{\mathsf{O}}(X) \to X$ in $\mathscr{D}$, where $\hat{\mathsf{O}}$ is the endofunctor

$$\hat{\mathsf{O}}\colon \mathscr{D} \to \mathscr{D}, \quad X \mapsto \mathsf{O} \cdot_{\mathcal{N}} X^{\star -} = \sum^{n \in \mathcal{N}} \mathsf{O}(n) \cdot X^{\star n},$$

where the right hand side is a coend. Joyal (I think) called this an *analytic* functor as it looks like a power series. In the world of algebraic operads, O would be called an $\mathbb{S}$-module and $\hat{\mathsf{O}}$ the associated Schur functor [16]. When O is an operad, then $\hat{\mathsf{O}}$ is a monad.

In the down to earth cases I care about this is an if and only if; even if it isn't, I find this approach is really nice as it works in greater generality and it incorporates 1.1.4.11 and 1.2.1.6 and many other ideas into the formalism in a neat way.

Example 1.2.3.2. For $\mathrm{O} = \mathrm{Ass}^{\mathbb{N}}$, $\hat{\mathrm{O}}(X)$ is the tensor algebra $\bigoplus_{n\geq 0} X^{\star n}$ which is the free associative algebra on X. For $\mathrm{O} = \mathrm{Comm}$, $\hat{\mathrm{O}}(X)$ is the free commutative algebra on X, that is, the symmetric algebra. The free Lie algebra is a more subtle topic, see e.g [22].

One can also carry the above over to a monoidal product on the functor category $[\mathcal{N}^{\mathrm{op}}, \mathscr{C}]$ (the *composite* or *substitution product* $\circ$). Just as groups are nowadays defined as abstract groups and not as transformation groups, I'd use this as definition:

Definition 1.2.3.3. An operad is a monoid in $([\mathcal{N}^{\mathrm{op}}, \mathscr{C}], \circ, \delta_0)$.

This has the advantage that we do not need to introduce any category $\mathscr{D}$ at all but study the operad in its own right. However, I am more interested in O-algebras, hence move on without explaining this approach in detail.

Example 1.2.3.4. In [21], Shulman indicates that you can go way beyond our setting: that an operation has a finite number of inputs is irrelevant and at least for $\mathscr{C} = \mathbf{Set}$ he claims you could actually also take $\mathcal{N} = \mathscr{C}^{\mathrm{op}}$. Then $[\mathcal{N}^{\mathrm{op}}, \mathscr{C}]$ is the category of endofunctors on $\mathscr{C}$ and $\circ$ becomes just the composition, so an operad is a monad on $\mathscr{C}$. This is a bit as with rings vs. algebras: rings are special $\mathbb{K}$-algebras (namely $\mathbb{K} = \mathbb{Z}$) and not the other way round.

1.2.4. *Approach V and VI.* Florian worked from the start with *multicategories* (= coloured operads). Then an operad is just a multicategory with a single object. I won't need this, but recall that to any monoidal category $\mathscr{D}$ Florian associated the multicategory $\mathrm{End}_{\mathscr{D}}$ *represented by* $\mathscr{D}$ (the endomorphism operad is the case of a monoidal category that is monoidally generated by one object). End is right adjoint to the *free monoidal category functor* F and allegedly (at least Tony, Zbiggi and Gemini seem to agree on this), the unit of this adjunction is an equivalence $\mathscr{D} \cong \mathrm{F}(\mathrm{End}_{\mathscr{D}})$: a monoidal category can be reconstructed from the associated (coloured) operad.

Finally, Florian told us how to associate to an operad O its *category of operators* $\mathrm{O}^{\otimes}$ which comes naturally with a Grothendieck opfibration whose codomain was in Florian's talk finite pointed sets.

MSc Topic 1.2.4.1. As far as I understand Shulman [21], the codomain is in general the category of operators $(\mathrm{End}_{\mathscr{N}})^{\otimes}$ of the operad associated to $\mathscr{N}$. For $\mathscr{N} = \mathbb{N}$, this is the opposite of the simplicial category Δ, so a planar operad can be characterised as a Grothendieck opfibration $\mathrm{O}^{\otimes} \to \Delta^{\mathrm{op}}$ (see Lurie's collected works, e.g. [17]). The advantage of this approach is that it is now relatively straightforward to replace categories by ∞-categories in order to define ∞-operads.

The construction of $\mathrm{O}^{\otimes}$ looks as if it could be the free monoidal category F(O), one forms forests and the fibre functor keeps track of their decompositions into trees (which in the symmetric case might be entangled and in the cartesian case even merge or diverge), but it seems things are more subtle: one first takes the free semicartesian operad on the given one, so one upgrades $\mathscr{N}$ if necessary (not sure what happens if that was already $\mathbb{F}$), and only then takes the free semicartesian monoidal category on this semicartesian operad. So for semicartesian multicategories, $\mathrm{F}(\mathrm{O}) = \mathrm{O}^{\otimes}$ as is also stated on nLab [18].

1.3. **Operads with multiplication.**

1.3.1. *Definition.*

Assumption 1.3.1.1. For the time being, $\mathscr{N} = \mathbb{N}$.

Definition 1.3.1.2. An *operad with multiplication* is an operad O together with an operad morphism $\mathrm{Ass}^{\mathbb{N}} \to \mathrm{O}$.

Remark 1.3.1.3. In $\mathscr{C} = \mathbf{Set}, \mathbf{Top}, \mathbf{Mod}_{\mathbb{K}}$, this is an operad together with an element $\mu \in \mathrm{O}_2$ such that $\mu \circ_{2,1} \mu = \mu \circ_{2,2} \mu$.

Remark 1.3.1.4. Gerstenhaber introduced this under the name *comp algebra* and called μ the *distinguished element*.

MSc Topic 1.3.1.5. As a continuation of 1.2.2.6, should one think of a pointed operad?

Remark 1.3.1.6. $\mathsf{Ass}^{\mathbb{N}} \to \mathsf{O}$ induces a forgetful functor from O-algebras to associative algebras. So O-algebras are associative algebras with more structure added.

1.3.2. *Examples.* The historic example is the following:

Example 1.3.2.1. Turning the endomorphism operad End_X into an operad with multiplication is the same as turning X into a monoid. This is the key example considered by Gerstenhaber in $\mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$ where X is simply a unital associative $\mathbb{K}$-algebra. However, recall the example $\mathscr{C} = \mathbf{Mod}_{\mathbb{K}}, \mathscr{D} = \mathbf{Mod}_{R^e}$ where R is a $\mathbb{K}$-algebra. Then X is a $\mathbb{K}$-algebra with a $\mathbb{K}$-algebra morphism $R \to X$ (an *R-ring*).

Example 1.3.2.2. If H is a Hopf algebra over $\mathbb{K}$ ($\mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$), define

$$\mathsf{C}(n) \coloneqq \mathbf{Mod}_{\mathbb{K}}(H^{\otimes n}, \mathbb{K})$$

and for each $\varphi \in \mathsf{C}(p)$ the map

$$D_\varphi \colon H^{\otimes p} \to H, \quad h^1 \otimes \cdots \otimes h^p \mapsto \varphi(h^1_{(1)}, \dots, h^p_{(1)}) h^1_{(2)} \cdots h^p_{(2)}.$$

Then C becomes an operad with multiplication

$$\mu = \varepsilon^H \mu^H$$

where $\mu^H \colon H \otimes H \to H$ is the multiplication in H and ε^H is its counit, and with

$$\begin{aligned} &(\varphi \circ_{p,i} \psi)(h^1, \dots, h^{p+q-1}) \\ \coloneqq\ &\varphi(h^1, \dots, h^{i-1}, D_\psi(h^i, \dots, h^{i+q-1}), h^{i+q}, \dots, h^{p+q-1}). \end{aligned}$$

Note that $\mu^H = D_\mu$. If A is a left H-comodule algebra, then generalising the formula for D_φ to

$$\alpha \colon \mathsf{C}(n) \to \mathsf{End}_A(n), \quad \alpha(\varphi)(a^1, \dots, a^n) \coloneqq \varphi(a^1_{(-1)}, \dots, a^n_{(-1)}) a^1_{(0)} \cdots a^n_{(0)}$$

turns A into a C-algebra. Somehow the associative operad gets twisted by C.

Remark 1.3.2.3. C is the $\mathbb{K}$-linear dual of the (unnormalised) bar construction of H, so this carries the structure of a cosimplicial $\mathbb{K}$-module whose cohomology is $\mathrm{Ext}_H(\mathbb{K}, \mathbb{K})$ and we will link this to the operad structure in a minute. One could also work directly with the bar construction and give it a *cooperad structure* which is maybe much more pleasing and enlightening in the context of bar-cobar duality. Note also all this extends to Hopf algebroids and then includes the case of the endomorphism operad covered by Gerstenhaber [11].

2. Gerstenhaber algebras up to homotopy

Here I explain how (algebraic) operads become (graded) pre-Lie algebras and operads with multiplication become Gerstenhaber algebras up to homotopy.

2.1. **Pre-Lie algebras.**

2.1.1. *Definition.*

Assumption 2.1.1.1. $\mathcal{N} = \mathbb{S}$, so $\mathcal{D}$ is a symmetric monoidal category and "operad" means "symmetric operad".

Remark 2.1.1.2. Any associative product $\bar{\circ}$ on a $\mathbb{K}$-module X turns X into a Lie algebra with respect to the commutator $[x, y] := x\bar{\circ}y - y\bar{\circ}x$, and like all such forgetful functors between types of algebra this can be expressed in terms of a morphism of symmetric operads $\mathsf{Lie} \to \mathsf{Ass}^{\mathbb{S}}$. We now generalise this.

Definition 2.1.1.3. A *pre-Lie* (aka *Vinberg*) *algebra* structure on $X \in \mathcal{D}$ is a binary operation $\bar{\circ}\colon X \star X \to X$ that satisfies

$$\alpha(x, y, z) = \alpha(x, z, y),$$

where $\alpha(x, y, z) := (x\bar{\circ}y)\bar{\circ}z - x\bar{\circ}(y\bar{\circ}z)$ is the *associator* of $\bar{\circ}$.

Remark 2.1.1.4. Obviously, there is a symmetric operad PreLie whose algebras are pre-Lie algebras. Just as with Lie, this only can be defined if $\mathcal{C}$ allows us to add (e.g. $\mathcal{C} = \mathbf{Mod}_{\mathbb{K}}$).

Proposition 2.1.1.5. *The commutator of a pre-Lie algebra structure is a Lie algebra structure.*

Proof. The skewsymmetry is given by definition, the question is whether the Jacobi identity is satisfied. Direct computation shows that

$$\begin{aligned}&[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]]\\ = &\sum_{\sigma \in S_3} (-1)^{|\sigma|} \alpha(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}),\end{aligned}$$

which vanishes if α is symmetric in the last two entries. □

Example 2.1.1.6. When $\mathscr{C} = \mathbf{GrMod}_{\mathbb{K}}$, “commutator” means “graded commutator”, so the true formula for the commutator without suppressed signs is $x \bar{\circ} y - (-1)^{|x||y|} y \bar{\circ} x$. Similarly, “Lie algebra” means “graded Lie algebra”, so

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

and there are also signs in the Jacobi identity

$$[x, [y, z]] + (-1)^{(|x|+|y|)|z|}[z, [x, y]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] = 0,$$

which can be neater written as

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[z, [x, y]] + (-1)^{|x||y|}[y, [z, x]] = 0.$$

Remark 2.1.1.7. The converse of 2.1.1.5 is not true. In fact, the proof shows that we also obtain a Lie bracket if $\alpha(x, y, z)$ is symmetric in x, y or if it is symmetric in x, z. In particular, if α is the associator of $\bar{\circ}$, then the associator of the opposite product $x \bar{\circ}^{\mathrm{op}} y = y \bar{\circ} x$ is given by $\beta(x, y, z) = \alpha(z, y, x)$. So $\bar{\circ}^{\mathrm{op}}$ is an example of the variation on pre-Lie algebras where one demands the associator to be symmetric in the first two entries. This is related to the convention choice whether pictures are read from left to right or from right to left as mentioned in 1.3.1.6. As yet another example, if $\bar{\circ}$ itself is a Lie bracket, then so is the its commutator, but here Jacobi tells us that the associator is rather symmetric in x and z.

2.1.2. *Examples.*

Example 2.1.2.1 (in honour of Julius)**.** Take $\mathscr{C} = \mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$. A *connection* on the vector fields on an affine scheme, that is, on the derivations $X := \mathrm{Der}_{\mathbb{K}}(A)$ of a commutative algebra $A \in \mathscr{D}$ (or more generally on a *Lie-Rinehart algebra* X over A) is a morphism

$$\nabla\colon\ X \otimes X \to X, \quad x \otimes y \mapsto \nabla_x y$$

such that $\nabla_{ax} y = a\nabla_x y$, $\nabla_x(ay) = x(a)y + a\nabla_x y$ holds for all $a \in A$ ($\nabla_x y$ is referred to as the covariant derivative of y along x with respect to the connection). The connection is *flat* if

$$\nabla_x(\nabla_y z) - \nabla_y(\nabla_x z) = \nabla_{[x,y]} z$$

and *torsionless* if

$$\nabla_x y - \nabla_y x = [x, y],$$

and one easily shows that $x\bar{\circ}y := \nabla_y x$ is a pre-Lie algebra structure if ∇ is flat and torsion-less.

Example 2.1.2.2. This is a warm-up for 2.1.2.6 below where I give a full proof. Take $\mathscr{C} = \mathscr{D} = \mathbf{Mod}_{\mathbb{K}}$, let T be the vector space with a basis given by all rooted trees (non-planer, no ordering of vertices), and define for trees x, y

$$x\bar{\circ}y := \sum_{i\in V(x)} x\circ_i y$$

where i runs through all vertices (not just leaves!) of x and $x\circ_i y$ is obtained by attaching the root of y to the vertex i as a new branch (as the branches of a vertex are not ordered, the question where to attach does not arise) . Then $(\mathsf{T}, \bar{\circ})$ is the free pre-Lie algebra $\widehat{\mathsf{PreLie}}(1)$ with a single generator.

Maybe the following only holds in characteristic 0, have forgotten:

Theorem 2.1.2.3 ([15])**.** *Let $H = \bigoplus_{i\geq 0}(X^{\otimes i})^{S_i}$ be the symmetric coalgebra on a vector space X (the coffee cocommutative conilpotent coalgebra on X that consists of all symmetric tensors over X with the deconcatenation coproduct). Then the pre-Lie algebra structures on X correspond bijectively to* right-sided *Hopf algebra structures on H, meaning those for which all $\bigoplus_{i\leq n}(X^{\otimes i})^{S_i}$, $n = 0, 1, 2, \dots$ are right ideals of H.*

Example 2.1.2.4. The Hopf algebra arising from rooted trees has been considered by Connes–Kreimer and has been generalised to a Hopf algebra whose (co)generators are labelled by Feynman diagrams. This yields a Hopf algebra approach to renormalisation.

MSc Topic 2.1.2.5. More a topic of Oliver: The Butcher group.

Proposition 2.1.2.6. *Every planar (i.e. nonsymmetric) operad* O *becomes a pre-Lie algebra with*

$$\varphi\bar{\circ}\psi := \sum_{i=1}^{p} \varphi\circ_{p,i}\psi, \quad \varphi \in \mathsf{O}(p).$$

Proof. The associator is given by

$$\alpha(\varphi, \psi, \chi) = \sum_{i=1}^{p}\sum_{j=1}^{p+q-1} (\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi - \sum_{j=1}^{q}\sum_{i=1}^{p} \varphi\circ_{p,i}(\psi\circ_{q,j}\chi)$$

$$
\begin{aligned}
&= \sum_{i=1}^{p}\sum_{j=1}^{i-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi + \sum_{i=1}^{p}\sum_{j=i}^{i+q-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi \\
&+ \sum_{i=1}^{p}\sum_{j=i+q}^{p+q-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi - \sum_{i=1}^{p}\sum_{j=1}^{q}\varphi\circ_{p,i}(\psi\circ_{q,j}\chi) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{i-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi + \sum_{i=1}^{p}\sum_{j=i+q}^{p+q-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi \\
&= \sum_{i=1}^{p}\sum_{j=1}^{i-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi + \sum_{i=1}^{p-1}\sum_{j=i+q}^{p+q-1}(\varphi\circ_{p,j-q+1}\chi)\circ_{p+q-1,i}\psi \\
&= \sum_{i=1}^{p}\sum_{j=1}^{i-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi + \sum_{i=1}^{p-1}\sum_{k=i+1}^{p}(\varphi\circ_{p,k}\chi)\circ_{p+r-1,i}\psi \\
&= \sum_{i=1}^{p}\sum_{j=1}^{i-1}(\varphi\circ_{p,i}\psi)\circ_{p+q-1,j}\chi + \sum_{k=1}^{p}\sum_{i=1}^{k-1}(\varphi\circ_{p,k}\chi)\circ_{p+r-1,i}\psi,
\end{aligned}
$$

which is manifestly symmetric in ψ, χ. Note α simply puts ψ and χ into all pairs of inputs of φ and sums up. □

Reappraisal 2.1.2.7. A planar operad is by very definition also a graded object in $\mathscr{C}$ with the arity of $\varphi \in \mathrm{O}(p)$ being its degree. However, the $\circ_{p,i}$ are not compatible with this degree; if we want them to be, we should rather consider the desuspension of O and hence define

$$|\varphi| := p - 1, \quad \varphi \in \mathrm{O}(p)$$

as we then have for $\varphi \in \mathrm{O}(p), \psi \in \mathrm{O}(q)$

$$|\varphi\circ_{p,i}\psi| = (p + q - 1) - 1 = p - 1 + q - 1 = |\varphi| + |\psi|.$$

So we associate to every operad $\mathrm{O} \in \mathscr{C}$ the graded presheaf $\bar{\mathrm{O}} \in [\mathscr{N}^{\mathrm{op}}, \mathbf{Gr}(\mathscr{C})]$ given by

$$\bar{\mathrm{O}}(n)(j) := \begin{cases} \mathrm{O}(n) & j = n - 1, \\ 0 & j \neq n - 1. \end{cases}$$

Is $\bar{\mathrm{O}}$ an operad in $\mathbf{Gr}(\mathscr{C})$? Almost, but the symmetry in the associativity axiom is the one from $\mathscr{C}$, not the one from $\mathbf{Gr}(\mathscr{C})$ that involves the sign $(-1)^{|\psi||\chi|}$ in cases 1 and 3. However, we can fix this (assuming that we have an operation $x \mapsto -x$ in $\mathscr{C}$ of course), by also redefining

$$\varphi \bar{\circ}_{p,i} \psi := (-1)^{|\psi|(i-1)} \varphi \circ_{p,i} \psi.$$

I memorise this sign by keeping in mind that when evaluating $\varphi \circ_{p,i} \psi$ on $x_1 \otimes \cdots \otimes x_{p+q-1}$, the ψ has to first jump over $i-1$ tensor components. And with this sign added, the associativity rules incorporate the correct signs, which, if I do not suppress them, read

$$(\varphi \bar{\circ}_{p,i} \psi) \bar{\circ}_{p+q-1,j} \chi = \begin{cases} (-1)^{(q-1)(r-1)} (\varphi \bar{\circ}_{p,j} \chi) \bar{\circ}_{p+r-1,i+r-1} \psi & j < i, \\ \varphi \bar{\circ}_{p,i} (\psi \bar{\circ}_{q,j-i+1} \chi) & i \leq j \leq i+q+1, \\ (-1)^{(q-1)(r-1)} (\varphi \bar{\circ}_{p,j-q+1} \chi) \bar{\circ}_{p+r-1,i} \psi & i+q \leq j. \end{cases}$$

Long story short:

Proposition 2.1.2.8. *If* $(\mathsf{O}, \circ_{p,i})$ *is a planar operad in* $\mathscr{C}$*, then* $(\bar{\mathsf{O}}, \bar{\circ}_{p,i})$ *is a planar operad in* $\mathbf{Gr}(\mathscr{C})$*.*

Remark 2.1.2.9. This degree shift is initially confusing, but can be seen in many other constructions. For example, when taking a projective resolution P of an object X in an abelian category $\mathscr{D}$, we have an exact sequence

$$0 \to P_d \to \ldots \to P_0 \to X \to 0$$

where P appears to be shifted in degree by 1. This is how the operadic degree shift sneaks in when one takes the Yoneda approach to Gerstenhaber algebras [19, 20, 6].

Corollary 2.1.2.10. *If* O *is a planar operad in* $\mathscr{C}$*, then* $\bar{\mathsf{O}}$ *is a pre-Lie algebra in* $\mathbf{Gr}(\mathscr{C})$ *via*

$$\varphi \bar{\circ} \psi := \sum_{i=1}^{p} \varphi \bar{\circ}_{p,i} \psi.$$

Definition 2.1.2.11. We denote the induced (graded!) Lie bracket on $\bar{\mathsf{O}}$ by $\{-,-\}$ and call this the *Gerstenhaber bracket*.

Example 2.1.2.12. If $\mu \in \mathsf{O}(2)$ is a binary operation, then

$$\mu \bar{\circ} \mu = \mu \circ_{2,1} \mu - \mu \circ_{2,2} \mu,$$

so μ is associative iff it is self-parallel with respect to the "connection" given by $\bar{\circ}$.

Remark 2.1.2.13. In [11] we have worked with $\bar{\circ}^{\mathrm{op}}$, hence there is a sign $(-1)^{|\varphi||\psi|}$ in front of the definition.

Remark 2.1.2.14. If one wants to avoid the introduction of $\bar{\mathsf{O}}$, one can view O as a graded Lie algebra whose Lie bracket is a morphism of degree 1, $\{-,-\} \in \mathrm{hom}_{\mathbf{Gr}(\mathscr{C})}(\mathsf{O} \otimes \mathsf{O}, \mathsf{O})(1)$.

2.2. **Brace algebras.**

2.2.1. *Gerstenhaber algebras up to homotopy.*

Assumption 2.2.1.1. O is a planar operad with multiplication.

MSc Topic 2.2.1.2. What is actually the right assumption about $\mathscr{C}$ here? I though the involution $x \mapsto -x$ is like a ribbon category, only symmetric, and $\mathbf{Set}$ also has that, with $-x = x$. But for pre-Lie algebra we need to add. If we demand $\mathscr{C}$ to be additive this is still not right, as we want to add elements of objects in $\mathscr{C}$. Maybe the best is not to have $\mathscr{C}$ at all but just an additive category $\mathscr{D}$?

Definition 2.2.1.3. The *cup product* is the binary operation

$$\mathsf{O}(p) \otimes \mathsf{O}(q) \to \mathsf{O}(p+q), \quad \varphi \smile \psi := (\mu \circ_{p,1} \varphi) \circ_{p+1,p+1} \psi$$

The *coboundary map* on O is given by

$$d := \{-, \mu\} \colon \mathsf{O}(n) \to \mathsf{O}(n+1).$$

Example 2.2.1.4. If $\mathsf{C}(n) = \mathbf{Mod}_{\mathbb{K}}(H^{\otimes n}, \mathbb{K})$ is the operad associated to a Hopf algebra H (Example 1.3.2.2), then

$$\begin{aligned}
&(\varphi \smile \psi)(h^1, \dots, h^{p+q}) \\
&= \varphi(h^1_{(1)}, \dots, h^p_{(1)})\psi(h^{p+1}_{(1)}, \dots, h^{p+q}_{(1)}\mu(h^1_{(2)} \cdots, h^p_{(2)}, h^{p+1}_{(2)} \cdots h^{p+q}_{(2)}) \\
&= \varphi(h^1_{(1)}, \dots, h^p_{(1)})\psi(h^{p+1}_{(1)}, \dots, h^{p+q}_{(1)}\varepsilon^H(h^1_{(2)} \cdots h^{p+q}_{(2)}) \\
&= \varphi(h^1, \dots, h^p)\psi(h^{p+1}, \dots, h^{p+q})
\end{aligned}$$

is the product dual to the *deconcatenation coproduct* on the bar construction [16] (unnormalised, meaning we do not just consider the tensor coalgebra of the augmentation ideal $\ker \varepsilon$ but of all of H). For $\varphi \in \mathsf{O}(p)$,

$$\begin{aligned}
&(\mu \bar{\circ} \varphi)(h^1, \dots, h^{p+1}) \\
&= \varphi(h^1, \dots, h^p)\varepsilon^H(h^{p+1}) + (-1)^{p-1}\varepsilon^H(h^1)\varphi(h^2, \dots, h^{p+1})
\end{aligned}$$

and

$$(\varphi\bar{\circ}\mu)(h^1,\dots,h^{p+1}) = \varphi(h^1h^2,h^3,\dots,h^p) \\ -\varphi(h^1,h^2h^3,h^4,\dots,h^p) \\ +\dots+(-1)^{p-1}\varphi(h^1,\dots,h^ph^{p+1}).$$

Thus

$$d\varphi = \{\varphi,\mu\} = \varphi\bar{\circ}\mu - (-1)^{p-1}\mu\bar{\circ}\varphi$$

is the dual of the boundary map of the bar construction,

$$(d\varphi)(h^1,\dots,h^{p+1}) = -(\varepsilon^H(h^1)\varphi(h^2,\dots,h^{p+1})- \\ \varphi(h^1h^2,h^3,\dots,h^{p+1})+\dots)$$

In particular, $d\circ d = 0$ and the cohomology of the cochain complex (O, d) is $\mathrm{Ext}_H(\mathbb{K},\mathbb{K}) \in \mathbf{GrMod}_{\mathbb{K}}$.

We now return to the general case of a planar operad and show that the above example is paradigmatic. The following lemmata show that $d, \smile$ and $\{-,-\}$ together equip O with a highly nontrivial algebraic structure.

Lemma 2.2.1.5. $dd = 0$.

Proof.

$$\{\{\varphi,\mu\},\mu\} = \{\varphi\bar{\circ}\mu - (-1)^{|\varphi|}\mu\bar{\circ}\varphi,\mu\} \\ = (\varphi\bar{\circ}\mu - (-1)^{|\varphi|}\mu\bar{\circ}\varphi)\bar{\circ}\mu \\ -(-1)^{|\varphi|+1}\mu\bar{\circ}(\varphi\bar{\circ}\mu - (-1)^{|\varphi|}\mu\bar{\circ}\varphi)$$

Recall that $\mu\bar{\circ}\mu = 0$, so

$$(\varphi\bar{\circ}\mu)\bar{\circ}\mu = (\varphi\bar{\circ}\mu)\bar{\circ}\mu - \varphi\bar{\circ}(\mu\bar{\circ}\mu) = \alpha(\varphi,\mu,\mu),$$

and this vanishes as the associator is symmetric in the last two entries and μ has degree $|\mu| = 1$ so symmetric means antisymmetric in this case. Similarly, the remaining three terms plus $0 = (\mu\bar{\circ}\mu)\bar{\circ}\varphi$ can be rewritten as

$$\alpha(\mu,\mu,\varphi) - (-1)^{|\varphi|}\alpha(\mu,\varphi,\mu)$$

which also vanishes in view of the symmetry of α. □

Lemma 2.2.1.6. *We have*

$$d(\varphi\bar{\circ}\psi) - \varphi\bar{\circ}d\psi - (-1)^{|\psi|}d\varphi\bar{\circ}\psi \\ = (-1)^{|\varphi|+|\psi|}(\varphi\smile\psi - (-1)^{|\varphi||\psi|}\psi\smile\varphi).$$

Proof. When we suppress signs, we have

$$\begin{aligned} d(\varphi\bar{\circ}\psi) &= (\varphi\bar{\circ}\psi)\bar{\circ}\mu - \mu\bar{\circ}(\varphi\bar{\circ}\psi), \\ (d\varphi)\bar{\circ}\psi &= (\varphi\bar{\circ}\mu)\bar{\circ}\psi - (\mu\bar{\circ}\varphi)\bar{\circ}\psi, \\ \varphi\bar{\circ}d\psi &= \varphi\bar{\circ}(\psi\bar{\circ}\mu) - \varphi\bar{\circ}(\mu\bar{\circ}\psi), \end{aligned}$$

so

$$\begin{aligned} &d(\varphi\bar{\circ}\psi) - \varphi\bar{\circ}d\psi - d\varphi\bar{\circ}\psi \\ &\quad = \alpha(\varphi,\psi,\mu) - \alpha(\varphi,\mu,\psi) + \alpha(\mu,\varphi,\psi). \end{aligned}$$

Note, however, that with the signs in, we'd rather get

$$\begin{aligned} &d(\varphi\bar{\circ}\psi) - \varphi\bar{\circ}d\psi - (-1)^{|\psi|}d\varphi\bar{\circ}\psi \\ &\quad = \alpha(\varphi,\psi,\mu) - (-1)^{|\psi|}\alpha(\varphi,\mu,\psi) + (-1)^{|\varphi|+|\psi|}\alpha(\mu,\varphi,\psi). \end{aligned}$$

The first line tells us that we should rather think of d as acting from the right; the second line is consistent as an expression evaluated on $\varphi\otimes\psi\otimes\mu$. That the associator $\alpha(\mu,\varphi,\psi)$ is simply the (graded) commutator of the cup product follows from the abstract formula (in the proof of Proposition 2.1.2.6). □

Remark 2.2.1.7. This is a version of the Eckmann–Hilton argument. One way to look at this is that if $\smile$ is graded commutative, then $\bar{\circ}$ descends to the cohomology of d. Another way to look at it is that on that cohomology, $\smile$ becomes graded commutative. Note that $\smile$ does not have to do the desuspension, it uses the arity as the degree.

Corollary 2.2.1.8. *$(\bar{O}, \{-,-\}, d)$ is a DG Lie algebra,*

$$d\{\varphi,\psi\} - \{\varphi, d\psi\} - (-1)^{|\psi|}\{d\varphi,\psi\} = 0.$$

Proof. The previous lemma tells us that $\bar{O}$ is in general *not* a DG pre-Lie algebra, but when taking the commutator, the right hand sides cancel out. □

Lemma 2.2.1.9. *We have*

$$(\varphi\smile\psi)\bar{\circ}\chi - (\varphi\bar{\circ}\chi)\smile\psi - (-1)^{(|\varphi|+1)|\chi|}\varphi\smile(\psi\bar{\circ}\chi) = 0.$$

Proof. Straightforward. Maybe one should number things from right to left... □

The other way round is much less pleasant:

Lemma 2.2.1.10. *Given $\varphi \in \mathsf{O}(p), \psi \in \mathsf{O}(q), \chi \in \mathsf{O}(r)$, define*

$$B_2(\chi,\varphi,\psi) \coloneqq \sum_{i=1}^{r} \sum_{j=i+p}^{p+r-1} (\chi \bar{\circ}_i \varphi) \bar{\circ}_j \psi.$$

Then

$$\begin{aligned} &\chi \bar{\circ} (\varphi \smile \psi) - (-1)^{|\chi||\psi|} (\chi \bar{\circ} \varphi) \smile \psi - (-1)^{|\chi|} \varphi \smile (\chi \bar{\circ} \psi) \\ = &\pm d B_2(\chi,\varphi,\psi) \pm B_2(d\chi,\varphi,\psi) \\ &\pm B_2(\chi, d\varphi, \psi) \pm B_2(\chi,\varphi, d\psi). \end{aligned}$$

MSc Topic 2.2.1.11. Find the signs using my $\bar{\circ}_i$ instead of the usual $\circ_i$. There are less terms than one thinks. I am not just not working them out here because I am too lazy but because below I will discuss brace algebras and then there will be a more conceptual ansatz for how to find them.

Remark 2.2.1.12. This is the one formula that Gerstenhaber did not find in full generality but only for cocycles.

Corollary 2.2.1.13. *We have*

$$\{\varphi \smile \psi, \chi\} - \{\varphi, \chi\} \smile \psi - (-1)^{(|\varphi|+1)|\chi|} \varphi \smile \{\psi, \chi\} = \dots$$

Remark 2.2.1.14. When it comes to $\smile$, $\{-,\chi\}$ thus behaves like a graded derivation acting from the left.

Corollary 2.2.1.15. $(\mathsf{O}, \smile, d)$ *is a DG algebra.*

Proof. That $\smile$ is associative on the nose is straightforwardly verified. The Leibniz rule follows from the previous corollary by taking $\chi = \mu$. □

Corollary 2.2.1.16. *The cohomology*

$$\mathsf{G} \coloneqq \ker d / \operatorname{im} d \in \mathbf{Gr}(\mathscr{C})$$

of an operad with multiplication is a Gerstenhaber algebra*, that is,*

$$\smile : \mathsf{G}(i) \otimes \mathsf{G}(j) \to \mathsf{G}(i+j)$$

turns G *into a graded commutative algebra,*

$$\{-,-\} : \mathsf{G}(i) \otimes \mathsf{G}(j) \to \mathsf{G}(i+j-1)$$

turns $\mathsf{s}^{-1}\mathsf{G}$ *into a graded Lie algebra, and we have*

$$\{\varphi \smile \psi, \chi\} = \{\varphi, \chi\} \smile \psi + (-1)^{(|\varphi|+1)|\chi|} \varphi \smile \{\psi, \chi\}.$$

2.2.2. *Brace algebras.* We have proved some lemmata and then stated a corollary, but what was the theorem? It was that the operad itself, not its cohomology, can be equipped with a strange algebraic structure:

Theorem 2.2.2.1. *An operad is naturally a brace algebra.*

Definition 2.2.2.2. A brace algebra structure on X is a sequence of $n+1$-ary operations

$$B_n\colon\ X^{\star n+1} \to X,$$

satisfying

$$\begin{aligned} &B_m(B_n(x, y_1, \dots, y_n), z_1, \dots, z_m) \\ &= \sum B_{i_0}(x, \dots, B_{i_1}(y_1, \dots), \dots, B_{i_n}(y_n, \dots), \dots), \end{aligned}$$

where all the $\dots$ are filled with the z_i in their original order.

Remark 2.2.2.3. Traditionally, one writes

$$B_n(x, y_1, \cdots, y_n) = x\{y_1, \cdots, y_n\}.$$

Remark 2.2.2.4. Of course here signs come in if one works in a graded context.

Example 2.2.2.5. When starting with an operad with multiplication, the 0-brace is the identity,

$$B_0 = \mathrm{id},$$

the pre-Lie product $\bar{\circ}$ is the 1-brace

$$\varphi\bar{\circ}\psi = B_1(\varphi, \psi) = \varphi\{\psi\}$$

and B_2 you have seen above, so the braces simply insert the n inputs $y_1, \dots, y_n$ in all ways into x. Note that for this the multiplication is not yet needed, but if there is one, it defines the cup product

$$\mu\{\varphi, \psi\}$$

and the coboundary map as before, and then one can derive all the formulas from the brace relations.

If the braces are symmetric in the y_i then they can be shown to be determined by the pre-Lie structure, so symmetric braces algberas are equivalent to pre-Lie algebras [14]. The following is thus the logical nonsymmetric version of 2.1.2.3:

Theorem 2.2.2.6 ([15])**.** *Let $H = \bigoplus_{i \geq 0} X^{\otimes i}$ be the tensor coalgebra on a vector space X (the cofree conilpotent coalgebra on X with the deconcatenation coproduct). Then the brace algebra structures on X correspond bijectively to right-sided Hopf algebra structures on H.*

3. Little discs

3.1. **More on Gerstenhaber algebras.**

3.1.1. *Poisson algebras.*

3.1.2. *Almost commutative algberas.*

3.1.3. *Lie–Rinehart algebras.*

3.1.4. *The recognition problem.*

3.1.5. *Deligne, Kontsevich & Co.*

3.2. **Littles discs and the Deligne conjecture.**

3.2.1. E_d.

3.2.2. $H(E_d)$.

3.2.3. $d = 1, d = 2$.

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